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Free convection stagnation point boundary layers driven by catalytic surface reactions: II Times to ignition

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Abstract. The steady states of a combustion model, derived in a previous paper, were shown to have critical points (turning points in the bifurcation diagram) for certain ranges of parameter values. Here attention is fixed on the heat release parameter λ and the time evolution for the solution for values of λ just above its critical value $\lambda_c^{(1)}$ is discussed. It is shown that the solution develops a three-stage structure, with the solution both approaching and leaving the critical point on a relatively short time scale. However, the majority of the time is spent in moving slowly past the critical point, on an $O((\lambda - \lambda_c^{(1)})^{-1/2})$ time scale. The solution finally attains its values on the upper solution branch, except in the special case of the exponential approximation and when reactant consumption is neglected. Here the temperature develops a singularity at a finite time t_B, of $O(\log(t_B - t))$, though the fluid velocity remains finite at t_B.

1. Introduction

In a previous paper [1] (part I of this series) we derived a model for the free convection boundary-layer flow near a forward stagnation point in which the heat input is generated by a reaction on the surface of a body. We took the reaction to be a single, first order exothermic one governed by Arrhenius kinetics. This resulted in equations for $F(y,t)$, $\theta(y,t)$ and $a(y,t)$, the dimensionless stream function, temperature and reactant concentration respectively, in the form

$$\frac{\partial^3 f}{\partial y^3} + \theta + f\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial f}{\partial y}\right)^2 = \frac{\partial^2 f}{\partial y \partial t}, \tag{1a}$$

$$\frac{1}{\sigma}\frac{\partial^2 \theta}{\partial y^2} + f\frac{\partial \theta}{\partial y} = \frac{\partial \theta}{\partial t}, \tag{1b}$$

$$\frac{1}{S_c}\frac{\partial^2 a}{\partial y^2} + f\frac{\partial a}{\partial y} = \frac{\partial a}{\partial t}. \tag{1c}$$

Here y measures distance normal to the body surface and t is time (both dimensionless) and σ and S_c are the Prandtl and Schmidt numbers respectively. The initial and boundary conditions to be satisfied are

$$f = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial \theta}{\partial y} = -\lambda a \exp(\theta/1 + \varepsilon\theta),$$

$$\frac{\partial a}{\partial y} = \alpha\lambda a \exp(\theta/1 + \varepsilon\theta) \quad \text{on} \quad y = 0 (t > 0) \tag{2a}$$

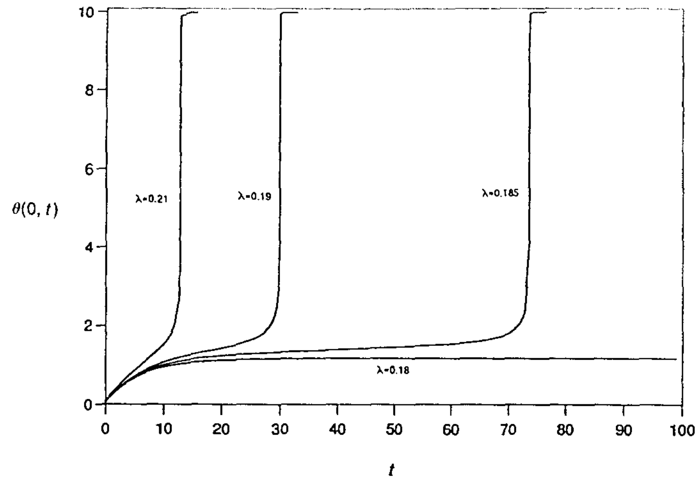

Fig. 1. Graphs showing how the wall temperature $\theta(0, t)$ evolves with t as λ is increased. Here $\lambda = 0.18, 0.185, 0.19, 0.21$ (with $\varepsilon = 0.0, \alpha = 0.1, \sigma = S_c = 1.0$).

$$\frac{\partial f}{\partial y} \to 0, \qquad \theta \to 0, \qquad a \to 1 \qquad \text{as } y \to \infty \qquad (t > 0), \tag{2b}$$

$$f = 0, \qquad \theta = 0, \qquad a = 1 \qquad \text{at } t = 0, \qquad (y > 0). \tag{2c}$$

The dimensionless parameters λ, ε and α are the measures of the heat released by the reaction, the activation energy and the rate of reactant consumption respectively and are as defined in part I as is a detailed derivation of the model.

In part I we considered the steady state solutions to equations (1, 2) in some detail showing that there are parameter ranges over which multiple solutions are possible (see the comment at the end of this section). The question that then arises is how does the solution evolve from the cold initial state, given by (2c) particularly as λ is increased through $\lambda_c^{(1)}$, its value at the lower critical point. This is illustrated in Fig. 1 where we show the time development of the solution for a range of values of λ (with $\varepsilon = 0.0$, $\alpha = 0.1$, $\sigma = S_c = 1.0$). For the values of $\lambda < \lambda_c^{(1)} = 0.1840$, in this case, $\theta(0, t)$ approaches the corresponding value on the lower steady state branch. The time for this stationary state to be reached increases as λ gets closer to $\lambda_c^{(1)}$. For values of $\lambda > \lambda_c^{(1)}$ there is no longer a steady state solution on the lower solution branch and the larger values on the upper solution branch are now the appropriate large time limit. Figure 1 shows that $\theta(0, t)$ does, in fact, approach these solutions as $t \to \infty$, with time taken to reach them getting progressively larger as $\lambda \to \lambda_c^{(1)}$ from above.

It is this latter aspect that we address in this paper, namely the so-called 'times to ignition' close to criticality. This problem arises generally in combustion and has been considered for somewhat different models by, for example, Boddington *et al.* [2, 3], Gray and Kordylewski [4] and Gray and Merkin [5]. These previous studies suggest that the time to ignition should increase, in our notation, like $(\lambda - \lambda_c^{(1)})^{-1/2}$ as $\lambda \to \lambda_c^{(1)}$; we find that this is the case in our boundary-layer model.

In all cases the final configuration of the system is the solution on the upper branch when $\lambda > \lambda_c^{(1)}$. The only exception is when both $\alpha = 0$ and $\varepsilon = 0$ as now there is no upper branch of solutions [1]. In this case we find that the solution has a finite time blowup at $t = t_B$ (say) with the wall temperature becoming infinite, of $O(\log(1/t_B - t))$, $t \to t_B$, though the fluid velocity remains finite at t_B.

Before proceeding with our discussion, we must mention that, since part I went to press, Professor A. Linan has pointed out to us that the bifurcation diagrams described in [1] can be obtained in a much simpler fashion. The details of this approach are given in the Appendix, which is due to him.

2. Solution of the initial value problem

We start our discussion of behaviour of initial-value problem (1, 2) close to criticality by taking

$$\lambda = \lambda_c^{(1)} + \delta, \qquad 0 < \delta \leqslant 1. \tag{3}$$

We will find that there are two time stages which we have to consider in detail. There is an initial stage, where t is of $O(1)$, and a further stage where t is of $O(\delta^{-1/2})$. We start by considering the initial development.

(I) Initial stage, t of $O(1)$

We look for a solution by expanding

$$\left.\begin{aligned} f(y,t;\delta) &= f_0(y,t) + \delta F_1(y,t) + \cdots \\ \theta(y,t;\delta) &= \theta_0(y,t) + \delta H_1(y,t) + \cdots \\ a(y,t;\delta) &= a_0(y,t) + \delta A_1(y,t) + \cdots \end{aligned}\right\}. \tag{4}$$

The equations satisfied by the leading terms (f_0, θ_0, a_0) are essentially equations (1), with boundary conditions (2a) modified appropriately.

Now, for the values of $\lambda < \lambda_c^{(1)}$, the solution of initial-value problem (1, 2) approaches the appropriate steady state solution on the lower branch, with the approach to this steady state being through exponentially small terms, of $O(e^{-\gamma t})$, γ is smallest eigenvalue and is real. The existence of a critical point at $\lambda = \lambda_c^{(1)}$ corresponds to saddle-node bifurcation and thus to the eigenvalue γ changing sign, with γ being zero at $\lambda = \lambda_c^{(1)}$. Consequently, the leading order solution (f_0, θ_0, a_0) approaches the corresponding steady state $(f_c(y), \theta_c(y), a_c(y))$ at the critical point with algebraic decay in t. In fact, we find that to obtain a consistent solution this decay must be $O(t^{-1})$. More precisely, we have

$$\left.\begin{aligned} f_0(y,t) &= f_c(y) + t^{-1} f_1(y) + t^{-2} f_2(y) + \cdots \\ \theta_0(y,t) &= \theta_c(y) + t^{-1} \theta_1(y) + t^{-2} \theta_2(y) + \cdots \\ a_0(y,t) &= a_c(y) + t^{-1} a_1(y) + t^{-2} a_2(y) + \cdots \end{aligned}\right\} \tag{5}$$

for t large.

At $O(t^{-1})$ we obtain a linear homogeneous system which has a nontrivial solution. In fact, it was the condition that this system had a nontrivial solution that was used previously in [1] to determine the critical point $\lambda_c^{(1)}$. However, the solution at this stage is determined only to within a constant multiple, with then, from [1],

$$f_1 = K_0 f_e, \qquad \theta_1' = K_0 \theta_e, \qquad a_1 = K_0 a_e \tag{6}$$

in terms of the eigenfunctions

$$f_e = yf_c' + f_c, \qquad \theta_e = y\theta_c' + 4\theta_c, \qquad a_e = ya_c' + 4a_c - 4$$

for any constant K_0.

The equations at $O(t^{-2})$ are, on using (6),

$$f_2''' + \theta_2 + f_c f_2'' - 2f_c' f_2' + f_c'' f_2 = K_0^2(f_e'^2 - f_e f_e'') - K_0 f_e', \tag{7a}$$

$$\frac{1}{\sigma}\theta_2'' + f_c\theta_2' + f_2\theta_c' = K_0^2 f_e\theta_e' - K_0\theta_e, \tag{7b}$$

$$\frac{1}{S_c}a_2'' + f_c a_2' + f_2 a_c' = K_0^2 f_e a_e' - K_0 a_e, \tag{7c}$$

subject to the usual boundary conditions and

$$\begin{aligned}\theta_2' &= -\lambda_c^{(1)} \exp\left(\frac{\theta_c}{1+\varepsilon\theta_c}\right) \\ &\quad \times \left[a_2 + \frac{K_0^2 a_e\theta_e}{(1+\varepsilon\theta_c)^2} + a_c\left(\frac{\theta_2}{1+\varepsilon\theta_c)^2} + \frac{K_0^2\theta_e^2(1-2\varepsilon-2\varepsilon^2\theta_c}{(1+\varepsilon\theta_c)^4}\right)\right], \\ a_2' &= \alpha\lambda_c^{(1)} \exp\left(\frac{\theta_c}{1+\varepsilon\theta_c}\right) \\ &\quad \times \left[a_2 + \frac{K_0^2 a_e\theta_e}{(1+\varepsilon\theta_c)^2} + a_c\left(\frac{\theta_c}{1+\varepsilon\theta_c)^2} + \frac{K_0^2\theta_e^2(1-2\varepsilon-2\varepsilon^2\theta_c}{(1+\varepsilon\theta_c)^4}\right)\right]\end{aligned} \tag{7d}$$

on $y = 0$.

To obtain a numerical solution of equations (7) we first construct two particular integrals (F_a, H_a, A_a) and (F_b, H_b, A_b). In both of these we put $F_i'(0) = 0$, $H_i(0) = 0$, $A_i(0) = 0$ $(i = a, b)$ and put $K_0 = 1$. For the first we include only the terms in K_0^2 in equations (7) with the corresponding values for $H_a'(0)$ and $A_a'(0)$ as given by (7d), while for the latter we include only the terms in K_0 in equations (7), with $H_b'(0) = A_b'(0) = 0$. We then construct three complementary functions (F_c, H_c, A_c), (F_d, H_d, A_d) and (F_e, H_e, A_e) in which we take $F_c''(0) = 1$, $H_c(0) = 0$, $A_c(0) = 0$; $F_d''(0) = 0$, $H_d(0) = 1$, $A_d(0) = 0$; $F_e''(0) = 0$, $H_e(0) = 0$, $A_e(0) = 1$. The corresponding values of $H_i'(0)$ and $A_i'(0)$ $(i = c, d, e)$ are then given by (7d) with K_0 put to zero. The complete solution of the problem is then given by

$$\left.\begin{aligned} f_2 &= K_0^2 F_a + K_0 F_b + cF_c + dF_d + eF_e \\ \theta_2 &= K_0^2 H_a + K_0 H_b + cH_c + dH_d + eH_e \\ a_2 &= K_0^2 A_a + K_0 A_b + cA_c + dA_d + eA_e \end{aligned}\right\} \tag{8a}$$

for some constants c, d, e.

From equations (7), we have, as $y \to \infty$,

$$F_i' \sim -\frac{C_i y}{c_0} + B_i, \qquad H_i \to C_i, \qquad A_i \to D_i \tag{8b}$$

for $i = a, b, c, d, e$ and where $c_0 = \lim_{y\to\infty} f_c(y)$. Thus to get a solution which satisfies all the outer boundary conditions, we must have

$$\left.\begin{aligned} cB_c + dB_d + eB_e &= -(K_0^2 B_a + K_0 B_b) \\ cC_c + dC_d + eC_e &= -(K_0^2 C_a + K_0 C_b) \\ cD_c + dD_d + eD_e &= -(K_0^2 D_a + K_0 D_b) \end{aligned}\right\}. \tag{8c}$$

Since equations (7) have a complementary function which satisfies the appropriate homogeneous boundary conditions, the matrix of coefficients in linear equations (8c) must be singular and these equations will then have a solution only if a compatibility condition is satisfied. This determines the value of the constant K_0 via, after a little calculation,

$$K_0 = \frac{[(D_eB_b - B_eD_b)(B_dC_e - B_eC_d) - (C_eB_b - B_eC_b)(B_dD_e - D_dB_e)]}{[(C_eB_a - B_eC_a)(B_dD_e - D_dB_e) - (D_eB_a - B_eD_a)(B_dC_e - B_eC_d)]}. \tag{8d}$$

Thus the solution at $O(t^{-1})$ is now fully determined.

We now consider the terms of $O(\delta)$ in expansion (4). These are given by linear equations, a consideration of which suggests that we should look for a solution, valid for t large, by expanding

$$\left.\begin{aligned} F_1(y,t) &= t\bar{f}_1(y) + \bar{f}_2(y) + \cdots \\ H_1(y,t) &= t\bar{h}_1(y) + \bar{h}_2(y) + \cdots \\ A_1(y,t) &= t\bar{a}_1(y) + \bar{a}_2(y) + \cdots \end{aligned}\right\}. \tag{9a}$$

The functions $(\bar{f}_1, \bar{h}_1, \bar{a}_1)$ are given by the same linear homogeneous system as (f_1, h_1, a_1) considered previously and have solution in terms of the eigenfunctions,

$$\bar{f}_1 = K_1 f_e, \qquad \bar{h}_1 = K_1\theta_e, \qquad \bar{a}_1 = K_1 a_e \tag{9b}$$

for some constant K_1.

When we consider the equations for the terms of $O(1)$ in expansion (9a) we find that they are essentially the same as equations (7) with the factor K_0^2 replaced by $2K_0K_1$, the factor K_0 by $-K_1$ and the boundary conditions at $y = 0$ becoming

$$\left.\begin{aligned} \bar{h}_2' &= -\exp\left(\frac{\theta_c}{1+\varepsilon\theta_c}\right)\left[a_c + \lambda_c^{(1)}\bar{a}_2 + \frac{2\lambda_c^{(1)}K_0K_1a_e\theta_e}{(1+\varepsilon\theta_c)^2}\right. \\ &\qquad \left. +\lambda_c^{(1)}a_c\left(\frac{\bar{h}_2}{(1+\varepsilon\theta_c)^2} + \frac{K_0K_1\theta_e^2(1-2\varepsilon-2\varepsilon^2\theta_c)}{(1+\varepsilon\theta_c)^2}\right)\right] \\ \bar{a}_2' &= \alpha\exp\left(\frac{\theta_c}{1+\varepsilon\theta_c}\right)\left[a_c + \lambda_c^{(1)}\bar{a}_2 + \frac{2\lambda_c^{(1)}K_0K_1a_e\theta_e}{(1+\varepsilon\theta_c)^2}\right. \\ &\qquad \left. +\lambda_c^{(1)}a_c\left(\frac{\bar{h}_2}{(1+\varepsilon\theta_c)^2} + \frac{K_0K_1\theta_e^2(1-2\varepsilon-2\varepsilon^2\theta_c)}{(1+\varepsilon\theta_c)^2}\right)\right] \end{aligned}\right\}. \tag{10}$$

To solve this system of equations numerically we use the complementary functions and particular integrals constructed previously and, in addition, we require a further particular integral (F_f, H_f, A_f) which is a solution of the homogeneous equations with

$$F_f''(0) = 0, \qquad H_f(0) = 0, \qquad A_f(0) = 0$$

and

$$H_f' = -a_c \exp\left(\frac{\theta_c}{1+\varepsilon\theta_c}\right), \qquad A_f' = \alpha a_c \exp\left(\frac{\theta_c}{1+\varepsilon\theta_c}\right) \quad \text{on } y=0.$$

The asymptotic behaviour of this particular integral is still given by (8b). The outer boundary conditions then require the solution of the linear equations

$$\left.\begin{aligned} cB_c + dB_d + eB_e &= -(2K_0K_1B_a - K_1B_b + B_f) \\ cC_c + dC_d + eC_e &= -(2K_0K_1C_a - K_1C_b + C_f) \\ cD_c + dD_d + eD_e &= -(2K_0K_1D_a - K_1D_b + D_f) \end{aligned}\right\}. \tag{11a}$$

Since, as noted previously, the matrix of coefficients of linear equations (11a) is singular, this system of equations will have a solution only if a compatibility condition is satisfied. This gives K_1 and, after a little calculation and on using (8d) for K_0, we obtain

$$K_1 = \frac{1}{3}\left[\frac{(B_fC_e - C_fB_e)(B_dD_e - D_dB_e) - (B_fD_e - D_fB_e)(B_dC_e - C_dB_e)}{(D_eB_b - B_eD_d)(B_dC_e - B_eC_d) - (C_eB_b - B_eC_b)(B_dD_e - D_dB_e)}\right] \tag{11b}$$

Thus we have, for t large,

$$\left.\begin{aligned} f_0(y,t) &\sim f_c(y) + \left(\frac{K_0}{t} + \delta K_1 t\right) f_e + \cdots \\ \theta_0(y,t) &\sim \theta_c(y) + \left(\frac{K_0}{t} + \delta K_1 t\right) \theta_e + \cdots \\ a_0(y,t) &\sim a_c(y) + \left(\frac{K_0}{t} + \delta K_1 t\right) a_e + \cdots \end{aligned}\right\}, \tag{12}$$

with K_0 and K_1 known. From (12) we can see that this expansion becomes non-uniform when t is $O(\delta^{-1/2})$ and a further stage is then required for t on this time scale.

(II) MIDDLE STAGE, t OF $O(\delta^{-1/2})$

In this region we introduce the long time scale

$$\tau = \delta^{1/2} t, \tag{13}$$

with (12) then suggesting that we look for a solution in this region by expanding

$$\left.\begin{aligned} f(y,\tau) &= f_c(y) + \delta^{1/2}\phi_1(y,\tau) + \delta\phi_2(y,\tau) + \cdots \\ \theta(y,\tau) &= \theta_c(y) + \delta^{1/2}h_1(y,\tau) + \delta h_2(y,\tau) + \cdots \\ a(y,\tau) &= a_c(y) + \delta^{1/2}g_1(y,\tau) + \delta g_2(y,\tau) + \cdots \end{aligned}\right\}. \tag{14}$$

When (13) and (14) are substituted into equations (1), we find that the leading order terms are simply the steady state equations at the critical point and are thus satisfied automatically. At $O(\delta^{1/2})$ we obtain a system of homogeneous linear equations and boundary conditions for (ϕ_1, h_1, g_1) (note that the time derivatives in the original equations (1) do not contribute at this stage). These equations have the solution

$$\phi_1 = \psi_1(\tau) f_e, \qquad h_1 = \psi_1(\tau)\phi_e, \qquad g_1 = \psi_1(\tau) a_e, \tag{15a}$$

where $\psi_1(\tau)$ is an, as yet, undetermined function of τ, with, on matching with (12),

$$\psi_1(\tau) \sim \frac{K_0}{\tau} + K_1\tau + \cdots \quad \text{as } \tau \to 0. \tag{15b}$$

At $O(\delta)$ we obtain the equations

$$\frac{\partial^3\phi_2}{\partial y^3} + h_2 + f_c\frac{\partial^2\phi_2}{\partial y^2} - 2f_c'\frac{\partial\phi_2}{\partial y} + f_c''\phi_2 = \left(\frac{\partial\phi_1}{\partial y}\right)^2 - \phi_1\frac{\partial^2\phi_1}{\partial y^2} + \frac{\partial^2\phi_1}{\partial y\partial\tau}, \tag{16a}$$

$$\frac{1}{\sigma}\frac{\partial^2 h_2}{\partial y^2} + f_c\frac{\partial h_2}{\partial y} + \phi_2\theta_c' = \frac{\partial\theta_1}{\partial\tau} - \phi_1\frac{\partial h_1}{\partial y}, \tag{16b}$$

$$\frac{1}{S_c}\frac{\partial^2 g_2}{\partial y^2} + f_c\frac{\partial g_2}{\partial y} + \phi_2 a_c' = \frac{\partial g_1}{\partial\tau} - \phi_1\frac{\partial g_1}{\partial y}. \tag{16c}$$

These equations are subject to the usual boundary conditions and, on $y = 0$,

$$\left.\begin{aligned}
\frac{\partial h_2}{\partial y} &= -\exp\left(\frac{\theta_c}{1+\varepsilon\theta_c}\right)\left[\lambda_c^{(1)}a_c\left(\frac{h_2}{(1+\varepsilon\theta_c)^2} + \frac{h_1^2(1-2\varepsilon-2\varepsilon^2\theta_c)}{2(1+\varepsilon\theta_c)^4}\right)\right. \\
&\qquad \left. + \frac{\lambda_c^{(1)}g_1h_1}{(1+\varepsilon\theta_c)^2} + \lambda_c^{(1)}g_2 + a_c\right] \\
\frac{\partial g_2}{\partial y} &= \alpha\exp\left(\frac{\theta_c}{1+\varepsilon\theta_c}\right)\left[\lambda_c^{(1)}a_c\left(\frac{h_2}{(1+\varepsilon\theta_c)^2} + \frac{h_1^2(1-2\varepsilon-2\varepsilon^2\theta_c)}{2(1+\varepsilon\theta_c)^4}\right)\right. \\
&\qquad \left. + \frac{\lambda_c^{(1)}g_1h_1}{(1+\varepsilon\theta_c)^2} + \lambda_c^{(1)}g_2 + a_c\right]
\end{aligned}\right\}. \tag{16d}$$

We look for a solution of equations (16) in the form

$$\phi_2 = \psi_2(\tau)\Phi_2(y), \qquad h_2 = \psi_2(\tau)H_2(y), \qquad g_2 = \psi_2(\tau)G_2(y). \tag{17}$$

When (15a) and (17) are substituted into equations (16) it results in non-homogeneous linear equations for Φ_2, H_2 and G_2 with right hand sides which involve terms in ψ_1^2/ψ_2 and ψ_1'/ψ_2, with boundary conditions (16d) giving non-homogeneous terms in ψ_1^2/ψ_2 and $1/\psi_2$. When we examine these equations in more detail we find that the terms in ψ_1^2/ψ_2 and ψ_1'/ψ_2 are essentially the same as the terms in K_0^2 and K_0 respectively in equations and boundary conditions (7) and the terms in $1/\psi_2$ in the boundary conditions are essentially the same as the terms independent of $\lambda_c^{(1)}$ in (10). Thus to obtain a solution we again use the previously constructed complementary functions and particular integrals. The outer boundary conditions require the solution for the linear equations

$$\left.\begin{aligned}
cB_c + dB_d + eB_e &= -\left(\frac{\psi_1^2}{\psi_2}B_a + \frac{\psi_1'}{\psi_2}B_b + \frac{1}{\psi_2}B_f\right) \\
cC_c + dC_d + eC_e &= -\left(\frac{\psi_1^2}{\psi_2}C_a + \frac{\psi_1'}{\psi_2}C_b + \frac{1}{\psi_2}C_f\right) \\
cD_c + dD_d + eD_e &= -\left(\frac{\psi_1^2}{\psi_2}D_a + \frac{\psi_1'}{\psi_2}D_b + \frac{1}{\psi_2}D_f\right)
\end{aligned}\right\}. \tag{18a}$$

Again the existence of a non-trivial solution to the related homogeneous problem means that the matrix of coefficients in linear equations (18a) is singular, and a solution is possible only if a compatability condition is satisfied. This gives, after some calculation, an equation for ψ_1, namely

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}\tau} = \frac{\psi_1^2}{K_0} + 3K_1. \tag{18b}$$

Equation (18b) is to be solved subject to matching condition (15b). Noting that, as the steady state solution at critical point is approached from below as $t \to \infty$ in the inner region, K_0 will be negative, the required solution is

$$\psi_1 = -\sqrt{3K_1|K_0|}\cot(\sqrt{3K_1/|K_0|}\tau). \tag{19}$$

The solution is still not complete, as (19) becomes singular at $\tau = \pi(|K_0|/3K_1)^{1/2}$ and to advance the solution we introduce a new time variable $\bar{t}$ given by

$$t = \pi\left(\frac{|K_0|}{3K_1}\right)^{1/2}\delta^{-1/2} + \bar{t}. \tag{20a}$$

This leaves equations (1) essentially unchanged (except that the differentiation is now with respect to $\bar{t}$ rather than t). The solution starts with [from (14), (15a) and (19)],

$$\left.\begin{aligned} f(y,\bar{t}) &\sim f_c(y) + \frac{|K_0|}{(-\bar{t})}f_e(y) + \cdots \\ \theta(y,\bar{t}) &\sim \theta_c(y) + \frac{|K_0|}{(-\bar{t})}\theta_e(y) + \cdots \\ a(y,t) &\sim a_c(y) + \frac{|K_0|}{(-\bar{t})}a_e(y) + \cdots \end{aligned}\right\}, \tag{20b}$$

as $\bar{t} \to -\infty$ and progresses on a $\bar{t}$ of $O(1)$ time scale to the steady state solution on the upper branch as $\bar{t} \to \infty$.

The behaviour of initial-value problem (1, 2) for values of λ just above the critical value $\lambda_c^{(1)}$ is now clear. There is an initial period in which the solution leaves its cold initial state and approaches the steady state values at criticality on an $O(1)$ time scale. There then follows a much longer period in which the solution very slowly passes the critical point, on a time scale of $((\lambda - \lambda_0^{(1)})^{-1/2})$. Having got past the critical point the solution is then free to rise very rapidly to its steady state solution on the upper branch. This three-time stage structure can be clearly seen in the results shown in Fig. 1.

As check on our theory, we calculated the times to ignition t_{ign}, from our numerical results. t_{ign} is not precisely defined and we decided to take t_{ign} to be that time at which the condition $\theta(0,t) > 4$ was first satisfied. Because the final rise to the upper branch is very rapid, other criteria for defining t_{ign} give essentially the same values (at least to within any numerical error). Now, since $t_{ign} \propto (\lambda - \lambda_c^{(1)})^{-1/2}$ it follows that

$$\lambda - \lambda_c^{(1)} \propto \frac{1}{{t_{ign}}^2} \tag{21}$$

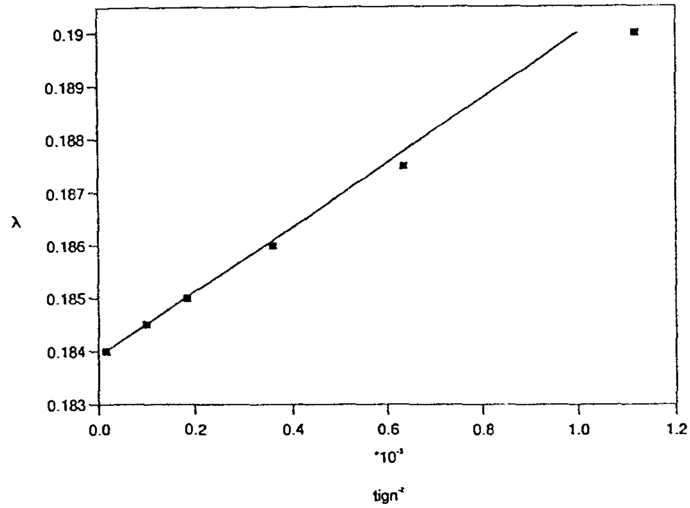

Fig. 2. Plots of λ against t_{ign}^{-2} for $\varepsilon = 0$, $\alpha = 0.1$, $\sigma = S_c = 1.0$. Values calculated from the numerical solution are shown by ■, the full line is the theoretical value.

and consequently a plot of λ against t_{ign}^{-2} should give a straight line. Results are shown in Fig. 2 (for the case $\varepsilon = 0.0$, $\alpha = 0.1$, $\sigma = S_c = 1.0$). The full line in Fig. 2 is the value obtained from the theory presented above for these values of parameters. From Fig. 2 we can see that the numerical results agree with the theoretical values as $\lambda \to \lambda_c^{(1)}$ which acts as confirmation of our theory.

The question that remains is the nature of this finite time blowup when $\varepsilon = 0$, $\alpha = 0$ for general values of $\lambda > \lambda_c^{(1)}$. This is what we discuss next.

3. The finite time blowup for $\varepsilon = \alpha = 0$, $\lambda > \lambda_c^{(1)}$

Here we consider general values of $\lambda > \lambda_c^{(1)}$ and take $\varepsilon = 0$, $\alpha = 0$, in which case $a(y, t) \equiv 1$ and we need consider only equations (1a, b) with boundary conditions (2a) becoming

$$\frac{\partial \theta}{\partial y} = -\lambda e^{\theta} \qquad \text{on } y = 0. \tag{22}$$

Suppose that the blowup occurs at a finite value t_B, which will depend upon λ. We put

$$\zeta = t_B - t \tag{23}$$

and look for a solution for $\zeta \ll 1$.

An examination of equations (1a, b) and boundary conditions (22) suggests that we start by putting

$$f = \zeta F, \qquad \theta = \frac{1}{2}\log(1/\zeta) + H, \qquad \eta = \frac{y}{\zeta^{1/2}}. \tag{24a}$$

We consider equation (1b) first, which becomes on using (23, 24a),

$$\frac{1}{\sigma}\frac{\partial^2 H}{\partial \eta^2} - \frac{1}{2}\eta\frac{\partial H}{\partial \eta} - \frac{1}{2} = -\zeta\frac{\partial H}{\partial \zeta} - \zeta^{3/2}F\frac{\partial H}{\partial \eta}, \tag{24b}$$

subject to the boundary condition

$$\frac{\partial H}{\partial \eta} = -\lambda e^{H} \qquad \text{on } \eta = 0 \tag{24c}$$

(the outer boundary conditions have to be relaxed at this stage).

Equation (24b) suggests looking for a solution by writing

$$H(\eta, \zeta) = H_0(\eta) + \zeta H_1(\eta, \zeta), \tag{25a}$$

where

$$H_0 = -\frac{\sigma}{2}\int_0^{\eta} e^{\sigma s^2/4}\int_s^{\infty} e^{-\sigma x^2/4}\,\mathrm{d}x\,\mathrm{d}s + b_0, \tag{25b}$$

where the constant b_0 is given by, from (24c),

$$b_0 = \log\left(\frac{\sqrt{\pi\sigma}}{2\lambda}\right) \tag{25c}$$

and, as $\eta \to \infty$,

$$H_0 \sim -\log\eta + b_0 + \cdots. \tag{25d}$$

We now turn to the equation for F. This is, on using (24a, 25a),

$$\frac{\partial^3 F}{\partial \eta^3} - \frac{\eta}{2}\frac{\partial^2 F}{\partial \eta^2} + \frac{1}{2}\frac{\partial F}{\partial \eta} = -\zeta\frac{\partial^2 F}{\partial \zeta \partial \eta} - \zeta^{3/2}\left(F\frac{\partial^2 F}{\partial \eta^2} - \left(\frac{\partial F}{\partial \eta}\right)^2\right) - \zeta^{1/2}\left(\frac{1}{2}\log(1/\zeta) + H_0 + \zeta H_1\right). \tag{26a}$$

Equation (26a) suggests looking for a solution by putting

$$F = a_0\eta^2 + \zeta^{1/2}\log(1/\zeta)\Phi(\eta, \zeta) \tag{26b}$$

for some constant a_0 and then expanding Φ in the form

$$\Phi(\eta, \zeta) = \phi_0(\eta) + \frac{1}{\log(1/\zeta)}\phi_1(\eta) + \cdots. \tag{27}$$

At leading order we obtain the equation

$$\phi_0''' - \frac{\eta}{2}\phi_0'' + \phi_0' = -\frac{1}{2}, \tag{28a}$$

which has solution satisfying $\phi_0(0) = \phi_0'(0) = 0$ and is not exponentially large as $\eta \to \infty$

$$\phi_0 = -\frac{\eta^3}{12}. \tag{28b}$$

At the next order we obtain

$$\phi_1''' - \frac{\eta}{2}\phi_1'' + \phi_1' = \phi_0' - H_0, \tag{29a}$$

subject to

$$\phi_1(0) = \phi_1'(0) = 0. \tag{29b}$$

The solution of equation (29a) can be written formally as

$$\phi_1' = -(\eta^2 - 2)\int_0^{\eta} \frac{e^{s^2/4}}{(s^2-2)^2}\int_s^{\infty}\left(\frac{x^2}{4} + H_0\right)(x^2-2)e^{-x^2/4}\,\mathrm{d}x\,\mathrm{d}s. \tag{30a}$$

As $\eta \to \infty$, we have

$$\phi_1 \sim \frac{\eta^3}{18}(3\log\eta - 1) + \left(b_0 - \frac{3}{2}\right)\eta + \cdots. \tag{30b}$$

When we come to consider the term of $O(\log^{-2}\zeta)$ in expansion (27b) we find that ϕ_2 satisfies a purely homogeneous equation, with the same left hand side as equation (28a), and thus the series terminates at $O(1/\log(1/\zeta))$. The next terms in the expansion of H and F are then both of $O(\zeta^{3/2})$. A further consideration of the equations for these terms of $O(\zeta^{3/2})$ shows that H and F are then $O(\eta)$ and $O(\eta^3)$ respectively for η large.

The solution given does not satisfy the outer boundary conditions and must be regarded as an inner solution. A further outer solution is required in which we leave all the variables unscaled. The solution in this region must satisfy the outer boundary conditions and match to the inner solution as $y \to 0$ (or $\eta \to \infty$ in the inner variables). From (25d) and (30a) we find that,

$$\left.\begin{aligned} \theta &\sim -\log y + b_0 + O(\zeta) \\ f &\sim a_0 y^2 + \frac{y^3}{18}(3\log y - 1) + O(\zeta) \end{aligned}\right\}, \tag{31a}$$

as $y \to 0$. (31a) suggests that in the outer region we expand

$$\left.\begin{aligned} f &= f_0(y) + \zeta f_1(y) + \cdots \\ \theta &= \theta_0(y) + \zeta\theta_1(y) + \cdots \end{aligned}\right\}, \tag{31b}$$

with the solution in the outer region remaining regular as $\zeta \to 0$. The leading order terms $f_0(y)$ and $\theta_0(y)$ are indeterminate, apart from being given by (31a) for y small and have $f_0' \to 0$, $\theta_0 \to 0$ as $y \to \infty$. The precise forms for $f_0(y)$ and $\theta_0(y)$ will depend on the values of λ and how the solution develops from its initial state at $t = 0$.

The nature of the singularity is now clear. There is a thin inner region in which the temperature has a logarithmic singularity though the velocity remains finite. There is also an outer region in which the flow is basically unaffected by the singularity in the temperature developing close to the surface. In more detail, from (24a, 26a) the skin friction remains finite with

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0} = a_0 + O(t_B - t) \quad \text{as } t \to t_B, \tag{32a}$$

where constant a_0 cannot be determined from the asymptotic expansion. However, the wall temperature becomes unbounded, with, from (24a, 25c),

$$\theta(0,t) \sim -\frac{1}{2}\log(t_B - t) + \log\left(\frac{\sqrt{\pi\sigma}}{2\lambda}\right) + \cdots \quad \text{as } t \to t_B. \tag{32b}$$

4. Conclusion

We have shown that when the heat release parameter λ is just above its critical value $\lambda_c^{(1)}$ the solution develops a three-stage time structure. The wall temperature (for example) both approaches and leaves its critical value on a short $O(1)$ time scale. However, the majority of the time is spent in moving slowly past the critical point, on the much longer $((\lambda - \lambda_c^{(1)})^{-1/2})$ time scale.

For the special case of no fuel consumption ($\alpha = 0$) and exponential approximation ($\varepsilon = 0$) the solution develops a singularity at a finite time, t_B, for all values of $\lambda > \lambda_c^{(1)}$ in which the wall temperature is of $O(\log(1/t_B - t))$ as $t \to t_B$ though the fluid velocity remains finite.

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The authors are greatly endebted to Professor A. Liñan for pointing out the very much simpler way in which the bifurcation diagrams, given originally in [1], can be obtained, as described in the Appendix.

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Appendix

The equations satisfied by the steady state $(f(y), \theta(y), a(y))$ are equations (1, 2) with the time derivatives put to zero. Suppose that $\theta(0) = \theta_0$, then the transformation

$$f = \theta_0^{1/4} F, \quad \theta = \theta_0 \phi, \quad a = 1 - (1 - a_0)\psi, \eta = \theta_0^{1/4} y, \tag{A1}$$

where $a_0 = a(0)$, results in the standard free convection problem

$$F''' + \phi + FF'' - F'^2 = 0, \tag{A2}$$

$$\phi'' + \sigma F \phi' = 0, \tag{A3}$$

$$\psi'' + S_c F \psi' = 0, \tag{A4}$$

subject to the boundary conditions

$$F(0) = F'(0) = 0, \quad \phi(0) = 1, \quad \psi(0) = 1, \quad F'(\infty) = \phi(\infty) = \psi(\infty) = 0. \tag{A5}$$

The solution of this problem then provides us with

$$\phi'(0) = -c_0(\sigma), \qquad \psi'(0) = -c_1(\sigma, S_c), \tag{A6}$$

where c_0 and c_1 are both positive. Note that when $S_c = \sigma$, $c_0 = c_1$.

If we now apply A1 and A6 to boundary conditions (2a) we find that

$$c_0 \theta_0^{5/4} = \lambda a_0 \exp(\theta_0/(1 + \varepsilon\theta_0)), \tag{A7}$$

$$c_1(1 - a_0)\theta_0^{1/4} = \alpha\lambda a_0 \exp(\theta_0/(1 + \varepsilon\theta_0)). \tag{A8}$$

Combining A7 and A8 gives

$$a_0 = 1 - \alpha\theta_0 c_0/c_1. \tag{A9}$$

Then, when A9 is applied in A7, we obtain finally

$$\frac{\lambda}{c_0} = \frac{\theta_0^{5/4}}{1 - \alpha\theta_0 c_0/c_1} \exp(-\theta_0/(1 + \varepsilon\theta_0)). \tag{A10}$$

Equation (A10) is a relation connecting $\theta_0 \equiv \theta(0)$ and λ and thus can be used to obtain all the bifurcation diagrams given in part I (plots of $\theta(0)$ against λ).

Two features of these bifurcation diagrams that were highlighted in part I were the limiting values of $\theta(0)$ on the upper solution branch and critical points (turning points on the bifurcation diagram). Both of these features can be deduced directly from A10. Clearly

$$\theta(0) \to \frac{c_1}{\alpha c_0} \quad \text{as } \lambda \to \infty. \tag{A11}$$

Previously, only the special case $S_c = \sigma$, where $\theta(0) \to 1/\alpha$, had been identified explicitly.

By differentiating A10, it is straightforward to show that at criticality

$$\gamma\varepsilon^2\theta_0^3 + (2\varepsilon\gamma - 4\gamma - 5\varepsilon^2)\theta_0^2 + (\gamma + 4 - 10\varepsilon)\theta_0 - 5 = 0, \tag{A12}$$

where $\gamma = \alpha c_0/c_1$. Expression A12 is the same as that derived in [1] for the special case $S_c = \sigma$, where $\gamma = \alpha$. Thus the discussion given in [1] for this special case can be extended directly to the general case $S_c \neq \sigma$.

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